ESC194

Aspen Erlandsson

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1 9.3 Separable Equations

$$\frac{dy}{dx} = F(x,y)$$
$$F(x,y) = g(x) \cdot f(y)$$

In other words, sometimes the function with x and y can be separated into a product of two different functions consisting exclusively of x or y. Or as a quotient:

$$\frac{g(x)}{h(x)}$$

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \to \int h(y)dy = \int = g(x)dx$$

$$h(y)\frac{dy}{dx} = g(x) \to \int h(y)\frac{dy}{dx} = \int g(x)dx$$

$$\to \frac{d}{dx}H(y) = h(y)$$

By chain rule, differentiate with respect to x now:

$$\frac{d}{dx}H(y) = h(y)\frac{dy}{dx}$$

$$\int h(y)\frac{dy}{dx} \int \frac{d}{dx}H(y)dx$$

Example:

$$\frac{dy}{dx} = \frac{1}{2}e^x y^2; y(0) = -1$$

$$\int \frac{2dy}{y^2} = \int e^x dx \to 2y^{-1} = e^x + c$$

or:
General solution:

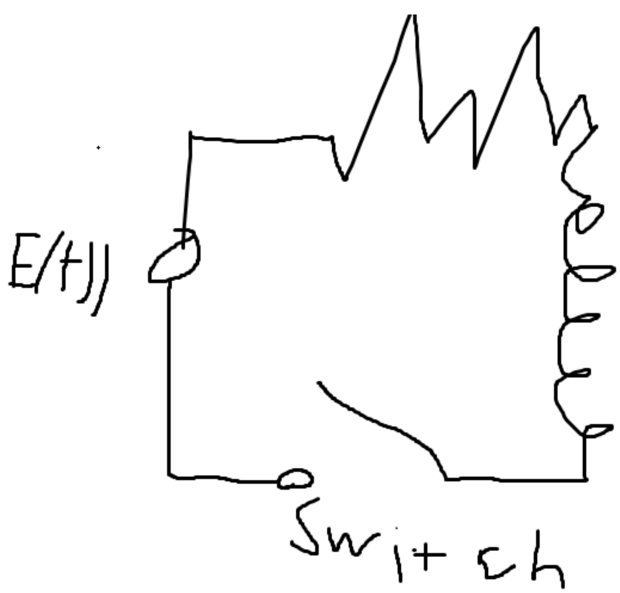
$$y = \frac{-2}{e^x + c}$$

at $x = 0; y = \frac{-2}{1+c} = -1$ we can find the particular solution:

$$\to c = 1$$

$$\therefore y = \frac{-2}{e^x + 1}$$

2 RL Circuits



$$V = RI \to \text{Ohm's Law}$$

$$V = L\frac{dI}{dt} \to \text{Inductor}$$

$$E(t) = L\frac{dI}{dt} + RI$$

Constant Voltage.

$$V = L\frac{dI}{dt} + RI$$

Close the switch at time = 0.

$$I(0) = 0$$

or:

$$\frac{dI}{dt} = \frac{V - RI}{L} \rightarrow dI \cdot \frac{L}{V - RI} = dt$$

$$V \neq RI$$

Example:

$$R = 10\Omega, L = 5H, V = 100V$$

$$\int \frac{dI \cdot 5}{100 - 10I} = \int dt$$

$$5(\frac{-1}{10}) ln\,|100 - 10I| = t + c$$

Don't need constants of integration on both sides, JUST ONE!

$$|100 - 10I| = e^{-2(t+c)} \rightarrow 100 - 10I = \pm e^{-2c} \cdot e^{-2t}$$

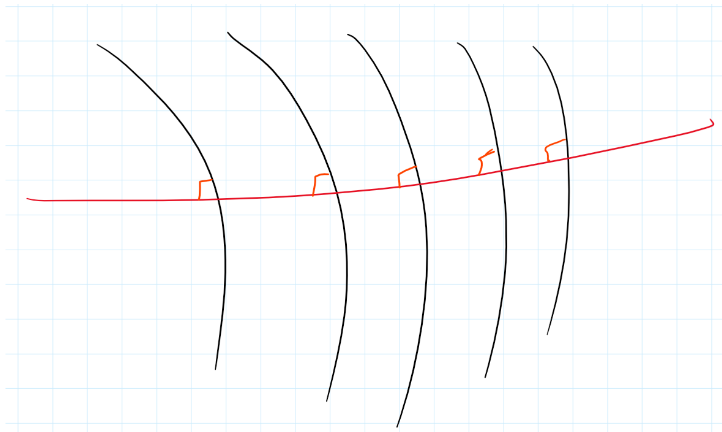
$\pm$ term is A, which is a constant.

$$I = 10 - \frac{A}{10}e^{-2t}$$

$$I(0) = 0 \rightarrow 0 = 10 - \frac{A}{10} \cdot 1 \rightarrow A = 100$$

$$\therefore I = 10(1 - e^{-2t})$$

3 Orthogonal Trajectories



Given that we have families of curves, can we find a curve that is orthogonal to every curve within a family at every intersection point?

$$f' = \frac{-1}{g'}$$

Example:

$$y^2 = kx^3 \rightarrow= 2yy' = 3kx^2$$

$$\rightarrow y' = \frac{3kx^2}{2y}$$

$$k = \frac{y^2}{x^3} \rightarrow y' = \frac{3(\frac{y^2}{x^3})x^2}{2y}$$

$$= \frac{3}{2}\frac{y}{x}$$

$$\therefore \text{orthogonal trajectory: } y' = \frac{-2x}{3y}$$

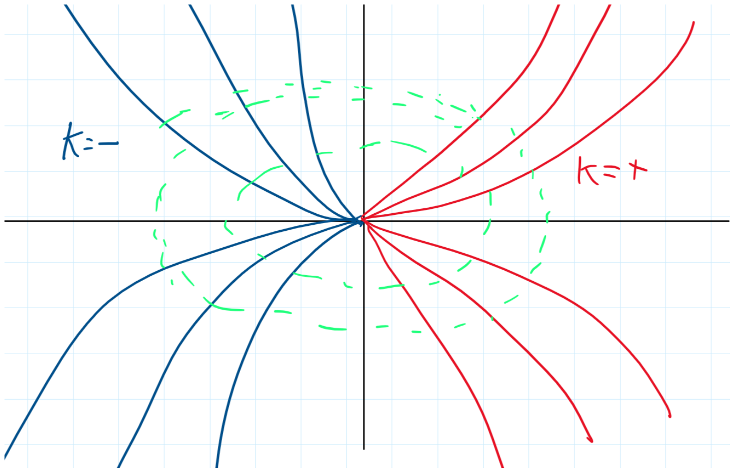
$$\rightarrow \int 3ydy = -\int 2xdx$$

$$\frac{3y^2}{2} = -x^2 + c$$

Multiplying through, we can rewrite as:

$$3y^2 + 2x^2 = 2c$$

Our family of curves is the series of ellipses defined above.



Green is our family of ellipses, blue and red show the original curves for positive and negative values of k respectively.

4 6.5 Exponential Growth and Decay

Rate of change is proportional to the amount of stuff we have.

$$\frac{df}{dt} = kf(t)$$

$$\rightarrow k = \frac{1}{f}\frac{df}{dt} = \frac{d}{dx}(ln(f))$$

$$\rightarrow ln(f) = kt + c^* \rightarrow f = e^{kt+c^*} = ce^{kt} \text{ where } c = e^{c^*}$$

$$c = f(t = 0) \rightarrow \text{initial value}$$

$$k = \text{growth constant(decay)}$$

Example:

Let's say each pair of parents can make 2.6 healthy rabbits per year, meaning that the growth is 1.3P per year.

$$\frac{dp}{dt} = 1.3P$$

$$P = 1000e^{1.3 \cdot 122} = 7.6 \cdot 10^{71}$$

Can we find doubling time?

$$2P_0 = O_0 e^{kt_2}$$
$$t_2 = \frac{ln(2)}{k}$$

Example: Earths population growth:

$$k = 0.011$$
$$\rightarrow t_2 = \frac{ln2}{12} = \text{ years}$$

5 Radioactive Decay

$$T \rightarrow^3 He + e^-$$

Tritium decays to helium plus an electron.

$$\frac{dN}{dt} = -kN$$

Written in this way so that k is positive, but overall the quantity is decreasing because the rate of change is negative.

$$\therefore N(t) = N_0 e^{-kt}$$
$$N(t_{1/2}) = \frac{1}{2}N(0) \rightarrow \frac{1}{2} = e^{-kt_{1/2}}$$
$$\rightarrow t_{1/2} = \frac{ln2}{k}$$

$t_{1/2}$ is the half life of radioactive material.

Example:
set $t = 0 \rightarrow 10$ years ago

$$\therefore 4 = N_0 e^{-k9}$$
$$3 = N_0 e^{-k10}$$

We can solve quite easily:

$$\frac{4}{3} = e^{-k(9-10)}$$
$$\rightarrow k = ln(\frac{4}{3}) \approx 0.288$$
$$N_0 = 4e^{k9} = 4e^{k9} = 4e^{2.59} = 53.3kg$$
$$t_{1/2} = 2.4\text{yrs}$$